Prediction of the Twisting Moment and Axial Force in a Circular Rubber Cylinder for Combined Extension and Torsion Based on the Logarithmic Strain Approach

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ABSTRACT: The scope of the present work is the application of a particular class of strain energy function, based on the logarithmic strain, for the prediction of the twisting moment and axial force of a rubber circular cylinder under combined extension and torsion. The strain energy function involves four material parameters three of which are determined by fitting published experimental data from simple tensile and compression tests of natural rubber. One of the parameters of the proposed model has physical meaning, and it is equal to one ninth of the initial modulus of elasticity of the material. Hence, the number of unknown parameters is reduced to three. The logarithmic strain energy function is then applied to a combined extension and torsion problem of a rubber circular cylinder to check its performance for more complicated deformations. The results are compared with corresponding experimental and theoretical solutions available in the literature to validate the proposed model. It is found that the proposed strain energy function apart from predicting the common modes of deformations is also capable to determine more complicated types of deformation. © 2008 Wiley Periodicals, Inc. J Appl Polym Sci 110: 1028–1033, 2008

Key words: logarithmic strain; torsion; strain energy function; rubber; logarithmic strain invariants

INTRODUCTION

The hyperelastic behavior of the rubber materials can be described using a special form of equation known as strain energy function from which the stress–strain law can be derived. For the designation of an appropriate strain energy function, for this kind of materials, phenomenological or macromolecular models may be developed. An extended review of these techniques is provided in a review article.[1] Several types of strain energy functions for rubber materials have been proposed in the past.[2–6]

Using phenomenological approaches, special forms of strain energy function have been developed based on the logarithmic strain approach. These energy functions lead to accurate estimations of the observed state of deformation-dependent behavior while their advantages on the study of large deformations are recognized.[7,8] The implementation of the logarithmic strain method in finite elements codes is complicated, which explains its relatively rare use in the literature, however a technique to be implemented has been recently proposed.[9] Anand[10,11] has proposed the Hencky's elasticity for solving hyperelastic problems. A practical way of exploiting the attractive properties of the logarithmic strain, which is used in the constitutive equations for the mechanical behavior of rubber like solids has been illustrated.[12–14] Diani and Gilormini[15] have combined the logarithmic strain and the full network model for a better understanding of the hyperelastic behavior of elastomers. Suitably defined invariants of the logarithmic strain have shown that are more adequate than the usual invariants of the left Cauchy-Green tensor to define the type of the constitutive equation of hyperelastic solids. Coupling these invariants with the macromolecular full network model, clarified some features of the state of strain dependence of these materials.[16] Plesek and Kruisova[17] have also studied the Hencky's elasticity model based on the logarithmic strain tensor.

Lately, a new strain energy function based on the logarithmic strain approach has been published,[7]

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which involves two unknown material parameters. This strain energy function is based on the strain invariants of the logarithmic strain and not on the invariants of the right and/or left Cauchy-Green strain tensors. In addition, the specific strain energy function has the advantage that the stress tensor includes also the logarithmic strain, whereas in some other forms[8] the logarithmic natural of the strain disappears after differentiation of the strain energy function for the evaluation of the stress tensor. The validity of the strain energy function based on the logarithmic strain approach was tested for various modes of deformation[7] such as simple tension, equibiaxial tension, and pure shear. As a further test of its good performance, it is clearly desirable to examine its consequences for other configurations in the light of what limited experimental data are available.

The purpose of this article is to investigate the predictions of an extended form of the logarithmic strain energy function, proposed in the aforementioned work[7] for some nonhomogeneous deformations. In the previous work,[7] the exponent of the second invariant of logarithmic strain was fixed, but here this exponent is used as an unknown parameter that must be determined by fitting. It is well known that the energy functions constructed on the basis of one-dimensional tests (such as simple tension) can often be unreliable in predicting data in other tests. Therefore, the proposed equation is used by utilizing the material parameters, which is derived by fitting both simple tension and compression (equivalent with equibiaxial deformation) experimental published data.[5] Firstly, the appropriate theory for the stress field, describing simple tension and compression is developed to evaluate the material parameters, which are incorporated in the proposed strain energy function. The theory is then extended for the prediction of twisting moment and axial force of a rubber cylinder under combined extension and torsion by using the estimated parameters. Analytical forms for the twist and axial force are derived and solved analytically by using a professional mathcode.[18]

LOGARITHMIC STRAIN ENERGY APPROACH FOR HYPERELASTIC SOLIDS

Logarithmic strain energy function

The Hencky's strain measure[19] taken as the logarithm of the extension ratio has well documented advantages, such as additivity of progressive elongation increments. Truesdell and Toupin[6] have discussed the history of the various logarithms measures and comment that, in effect, using an analytical continuation of the series for logarithms results in off-diagonal terms involving infinite series. An additively symmetrical measure of strain must satisfy the relation $e(1/\lambda) = -e(\lambda)$, where λ represents the one-dimensional stretch ratio, that is, the ratio of deformed to undeformed length.[19] One of the infinitely, many smooth functions that conform to this equation is

$$\mathrm{e} = \ln(\lambda) \tag{1}$$

This equation was generalized, to a proper tensorial representation, for finite deformation theory.[20] The polar decomposition of the deformation tensor **F** yields the right Cauchy-Green stretches tensor, U, whose principal values are defined by λ_i (i = 1–3). Assuming the existence of a scalar potential function of three independent invariants of e = ln(U), one can write the three logarithmic invariants of logarithmic strain:

$$I_e = \sum_{i=1}^{3} e_{\mathrm{i}}, \quad I_{ee} = \sum_{i=1}^{3} e_i^2, \quad I_{eee} = \sum_{i=1}^{3} e_i^3 \tag{2}$$

where $e_i = \ln(\lambda_i)$ (i = 1–3) defines the principal values of logarithmic strain **e**. For hyperelastic solids a stress measure **S**, can be derived from a scalar potential function $W = W(I_e, I_{ee}, I_{eee})$:

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{e}} = \sum_{k=1}^{3} \frac{\partial W}{\partial \mathrm{I}_k} \frac{\partial I_k}{\partial \mathbf{e}} \tag{3}$$

where the invariants I_i (i = 1–3) are: $I_1 = I_e$, $I_2 = I_{ee}$, $I_3 = I_{eee}$. One such form of the strain energy function W, based on the logarithmic strain measure, can be written as:

$$W(I_{ee}, I_{eee}) = hI_{eee} + gI_{ee} + kI_{ee}^{\alpha} \tag{4}$$

where the material constants h, k, α must be determined by fitting with the available experimental data in the literature. The parameter g is equal to the one ninth of the modulus of elasticity E, as it was proved earlier.[7] Notice that in this work, the exponent of the last term is not fixed in contrast with the previous work,[7] in which this exponent was taken equal to 3/2.

Application of the logarithmic strain energy function to simple tension and compression

For incompressible solids the product of the principal values λ_i is unity, hence the summation of the logarithmic stretches is zero, that is, the first invariant of the logarithmic strain is null, $I_e = 0$. The principal values of the Cauchy stress are given by[21,22]:

$$t_j = \lambda_j \frac{\partial W}{\partial \lambda_j} - p = \frac{\partial W}{\partial e_j} - p \tag{5}$$

where p represents the hydrostatic pressure, which is determined by satisfying the equilibrium and boundary conditions. Substituting eq. (2) into eq. (5), yields:

$$t_i = 3he_i^2 + 2\left\{ g + \alpha k \left(\sum_{i-1}^{3} e_i^2 \right)^{\alpha-1} \right\} e_i - p \tag{6a}$$

For simple tension tests the principal stretches are defined by $\{\lambda, \lambda^{-1/2}, \lambda^{-1/2}\}$, where λ is defined along the direction of force. The principal Cauchy stresses are defined by $\{t,0,0\}$, the principal logarithmic strains are $(e, -e/2, -e/2)$, and the second and third logarithmic strain invariants are $I_{ee} = (3/2)e^2$, $I_{eee} = (3/4)e^3$. Hence, for this type of deformation eq. (6a) can be written as:

$$t_{\mathrm{ST}}(e) = 9ge + \frac{27}{4}he^2 + 6\left(\frac{3}{2}\right)^{\alpha} ke^{2\alpha-} \tag{6b}$$

The Cauchy stress for this mode of deformation dependents on the material parameters h, k, a, and not on the parameter g, because $g = E/9$. This is due to the fact, that eq. (6b) must satisfy the Hooke's law for small strains.

For simple compression tests, that is equivalent to equibiaxial tension, the principal stretches are defined by $\{\lambda, \lambda, \lambda^{-2}\}$, where λ defines the stretch along the loading direction, and the principal true stresses are defined by $\{t, t, 0\}$. The principal logarithmic strains are defined by $\{e, e-2e\}$ and the strain invariants are $I_{ee} = 6e^2$, $I_{eee} = -6e^3$. The true stress, from eq. (6a), can be written as:

$$t_{\mathrm{sc}}(e) = 6g\ e - 9h\ e^2 + \alpha\ k6^{\alpha}\ e^{2\alpha-1} \tag{6c}$$

This mode of deformation is equivalent to equibiaxial tension tests and dependents also on the same parameters, as in the case of simple tension. In order the theory to work well for rubber materials, the unknown parameters must fit both types of deformation.

Application of the logarithmic strain energy function to combined extension and torsion of a circular cylinder

Following Ogden,[23,24] the deformation in a cylindrical coordinate system (r, θ, z) for the extension and torsion of a circular cylinder is defined by:

$$\lambda = z/Z, \quad r = R/\sqrt{\lambda}, \quad \theta = \Theta + \tau\lambda Z \tag{7a}$$

where the solid cylinder is considered incompressible, that is, is made from a rubber material, having radius R and length L in the reference configuration. The coordinates (R, Θ, Z) corresponds to the undeformed state, whereas the coordinates (r, θ, z) refers to the deformed system. The principal stretches λ_i $(i = 1–3)$ are defined by $\lambda_1 = \lambda^{-1/2}$, along the radial direction, whereas the other principal stretches are defined via the equations:

$$\lambda_2^2\lambda_3^2 = \lambda, \quad \lambda_2^2 + \lambda_3^2 = \lambda^{-1} + \lambda^2 + \lambda\tau^2R^2 \tag{7b}$$

where τ $(\tau > 0)$ defines the twist (or torque) per unit length of the deformed cylinder and R defines the radius of the undeformed cylinder. Introducing the further notation χ $(\chi \succ 1)$ eq. (7b) can be written as:

$$\lambda_2 = \lambda^{1/4}\chi, \quad \lambda_3 = \lambda^{1/4}\chi^{-1} \tag{8}$$

so that eq. (7b) becomes:

$$\chi^2 + \chi^2 = (m^2+1)\nu^{-1} + \nu \tag{9a}$$

or

$$m^2 = (\chi^2 - \nu)(\nu - \chi^{-2}) \tag{9b}$$

where

$$\nu = \lambda^{3/2} \text{ and } m = \tau r \nu \tag{9c}$$

The principal axis of strain in the deformed configuration associated with the greatest stretch has direction cosines $(0, \cos\psi, \sin\psi)$ relative to (r, θ, z) directions where:

$$\begin{aligned} \sin(2\psi) &= \frac{2m}{\chi^2 - \chi^{-2}}, \\ \cos(2\psi) &= \frac{(\chi^2 + \chi^{-2} - 2\nu)}{\chi^2 - \chi^{-2}} \quad \forall\left(0 \le \psi \le \frac{\pi}{4}\right) \end{aligned} \tag{10}$$

The torsional couple is defined by:

$$M = \int_0^R \sigma_{z\theta} 2\pi r R dR \tag{11}$$

where R defines the radius of the solid circular cylinder.

The shear stress $\sigma_{z\theta}$ is defined by:

$$\sigma_{z\theta} = \frac{1}{2}(t_2 - t_3)\sin(2\psi) \tag{12a}$$

where t_α $(\alpha = 1–3)$ are the principal Cauchy stresses defined in eq. (6a). Using eq. (6a), the principal stresses t_i $(i = 2,3)$, are replaced into eq. (12a) and the stress components $\sigma_{z\theta}$ become:

$$\sigma_{z\theta} = \left\{ 3h\ \ln(\lambda)c + 2\left(g + \alpha k\left(\frac{3}{8}\ln^2(\lambda) + 2\ \ln^2(\chi)\right)^{\alpha-1}\right)\right\} \frac{m\ \ln(\chi)}{\chi^2 - \chi^{-2}} \quad (12b)$$

By substituting eq. (12a) into eq. (12b) we obtain

$$M(\lambda; h; g; k; \alpha; \tau) = 2\pi\tau^{-3}\nu^{-2} \times \int_{\nu^{1/2}}^{\chi R} \left\{ 3h\ \ln(\lambda) + 2\left(g + \alpha k\left(\frac{3}{8}\ \ln^2(\lambda) + 2\ \ln^2(\chi)\right)^{\alpha-1}\right\} \times \frac{(\chi^2\ + \chi^{-2} - 2)\ln(\chi)}{\chi} d\chi \quad (12c)$$

For simple torsion ($\lambda = 1$), eq. (12c) is reduced to the following expression:

$$M = \pi\tau^{-3} \left\{ \begin{array}{l} 4g(\xi\ \sin h\xi - \cos h\xi - \frac{\xi^2}{2})\Big|_{\xi_l}^{\xi_u} \\ + 2^{1-\alpha}\alpha k \int_{\xi_l}^{\xi_u} \xi^{2\alpha-1}(\cos h\xi - 1)d\xi \end{array} \right\} \quad (13a)$$

where the lower and upper limits of integration are:

$$\xi_l = \frac{3}{2}\ \ln(\lambda) = 0 \quad (13b)$$

and

$$\xi_{11} = 2\ \ln\left[\frac{1}{2}\left(\sqrt{(\tau R)^2 + 4} + \tau R\right)\right] \quad (13c)$$

For $k = 0$ eq. (13a) leads to Anand's solution[10,11] for torsional couple. The axial force N, can be written as[23]:

$$N = \pi \int_0^{R/\sqrt{\lambda}} \{(t_{rr} - t_{\theta\theta}) + 2(t_{zz} - t_{rr})\} r dr \quad (14)$$

where

$$t_{rr} - t_{\theta\theta} = \frac{\partial W}{\partial e_1} - \frac{1}{2}\left(\frac{\partial W}{\partial e_2} + \frac{\partial W}{\partial e_3}\right) - \frac{1}{2}\left(\frac{\partial W}{\partial e_2} - \frac{\partial W}{\partial e_3}\right)\cos 2\psi \quad (15)$$

and the derivatives of the strain energy function $W(e_1, e_2, e_3)$ are given via eq. (6a) and

$$\cos(2\psi) = \frac{\chi^2 + \chi^{-2} - 2\nu}{\chi^2 - \chi^{-2}} \quad (16)$$

Substituting the derivatives of W and the $\cos(2\psi)$ into eq. (15) yields:

$$t_{rr} - t_{\theta\theta} = -3h\ \ln^2(\chi) - 2\left[\left(g + \alpha k^{2\alpha-1}\ \ln^{2\alpha-2}(\chi)\right)\ln(\chi)\right]\left(\frac{\chi^2 + \chi^{-2} - 2\nu}{\chi^2 - \chi^{-2}}\right) \quad (17)$$

The second term in eq. (14) is determined by:

$$t_{zz} - t_{rr} = -\frac{\partial W}{\partial e_1} + \frac{1}{2}\left(\frac{\partial W}{\partial e_2} + \frac{\partial W}{\partial e_3}\right) - \frac{1}{2}\left(\frac{\partial W}{\partial e_2} - \frac{\partial W}{\partial e_3}\right)\cos 2\psi \quad (18)$$

or equivalently

$$t_{zz} - t_{rr} = 3h\ \ln^2(\chi) - 2\left[\left(g + \alpha k^{2\alpha-1}\ \ln^{2\alpha-2}(\chi)\right)\ln(\chi)\right]\left(\frac{\chi^2 + \chi^{-2} - 2\nu}{\chi^2 - \chi^{-2}}\right) \quad (19)$$

Substituting eqs. (17) and (18) into eq. (14) yields:

$$N = \frac{\pi\nu^{-1}\tau^{-2}}{2}\Big\{ -\frac{9}{8}h\ \ln^2(\lambda)ch(\xi) + 3g\ \ln(\lambda)sh(\xi) + \frac{6h}{8}\int_{\xi_l}^{\xi_u} \xi^2(\exp(\xi) - \exp(-\xi))d\xi + \frac{9k}{8}\ln(\lambda)\int_{\xi_l}^{\xi_u} (\gamma_2^2 + 2\xi^2)^{1/2}(\exp(\xi) - \exp(-\xi))d\xi - \frac{3h}{4}\ln(\lambda)\int_{\xi_l}^{\xi_u} \xi(\exp(\xi) - \exp(-\xi) - 2\nu)d\xi + g\int_{\xi_l}^{\xi_u} (\gamma_2^2 + 2\xi^2)^{1/2}\xi(\exp(\xi) - \exp(-\xi) - 2\nu)d\xi\Big\} \quad (20a)$$

where $\gamma_2^2 = \frac{3}{2}\ln^2(\lambda)$ and the lower and upper limits of the integral were defined earlier. For simple torsion experiments, $\lambda = 1$, eq. (20a) yields:

$$N = 3\pi\tau^{-2} \left\{ \begin{array}{l} \frac{h}{2}\left[(\xi^2 + 2)\cos h\xi - 2\xi\sin h\xi\right]_{\xi_l}^{\xi_u} \\ - 4g\left(\xi\ \sin h\xi - \cos h\xi - \frac{\xi^2}{2}\right)_{\xi_l}^{\xi_u} \\ - ak2^{1-\alpha}\int_{\xi_l}^{\xi_u} \xi^{2\alpha-1}(\cos \mathrm{h}\xi - 1)d\xi \end{array} \right\} \quad (20b)$$

where $\xi = 2\ \ln(\chi)$, and the upper and lower limits of the integral were defined earlier in eqs. (13b,c). For $h = 0$ and $k = 0$ eq. (20) is reduced to Anand's solution.[10,11]

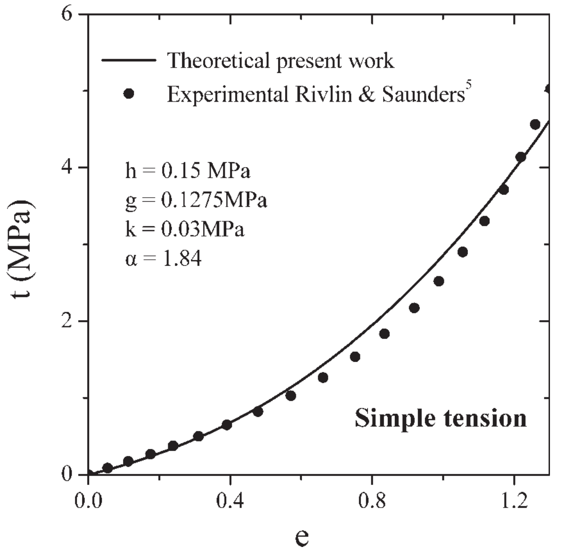

Figure 1 Fitting of the experimental data[5] with eq. (6a).

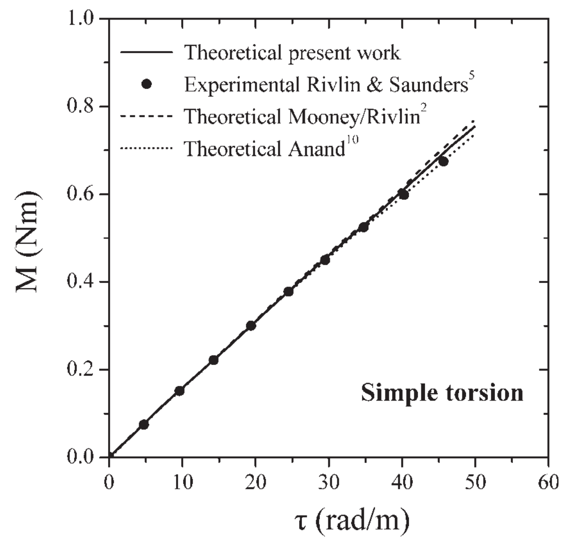

Figure 3 Prediction of the twisting moment with respect to the twisting angle per unit length.

RESULTS AND DISCUSSION

Simple tension and compression

From the experimental data[5] one can extract the ground modulus of the testing rubber material equal to $E = 1.1475$ MPa, hence the g value is equal to 0.1275 MPa. Equations (6b) and (6c) which apply for simple tension and compression, respectively, were used for the prediction of Rivlin's and Saunders[5] experimental data. The values of the material parameters were estimated $h = 0.15$ MPa, $k = 0.003$ MPa, and $\alpha = 1.84$ after a fitting process the results of which are illustrated in Figures 1 and 2 for the tensile and compression loading conditions, respectively. It is obvious that the proposed strain energy function can adequately describe the tensile and compressive behavior of the rubber for a large range of strains up to break. The present theory is an extension of Anand's approach that clearly fits the simple tension and compression data (Figs. 1, 2) for the rubber material nicely, in contrast to Anand's[10,11] constitutive equation that fits the experiments in the small region near $\lambda = 1$.

Combined extension and torsion of a solid circular cylinder

Equations (13) and (20) are solved using a computer math code,[18] and the results that arise are presented in Figures 3–5. A value of 12.7 mm was assigned for the radius, R, of the rubber cylinder to make

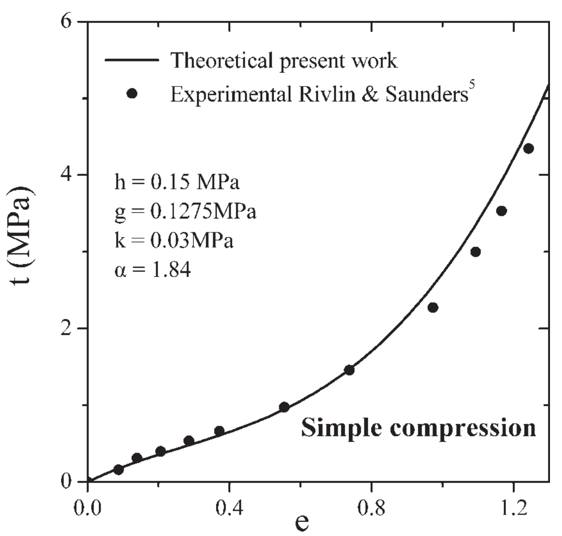

Figure 2 Fitting the experimental data[5] with eq. (6c).

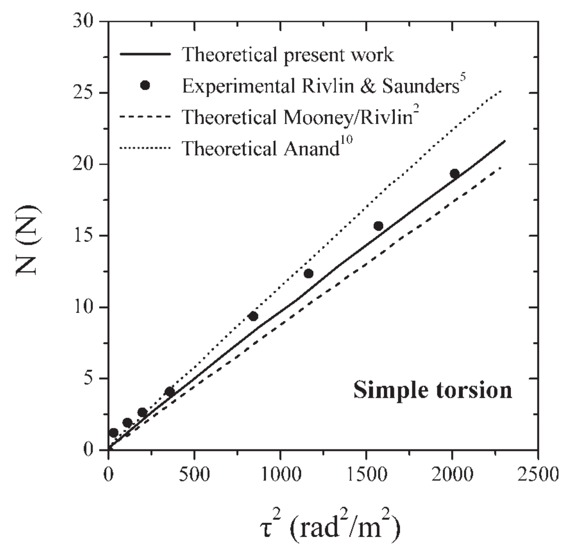

Figure 4 Prediction of the axial force with respect to the square of the twisting angle per unit length.

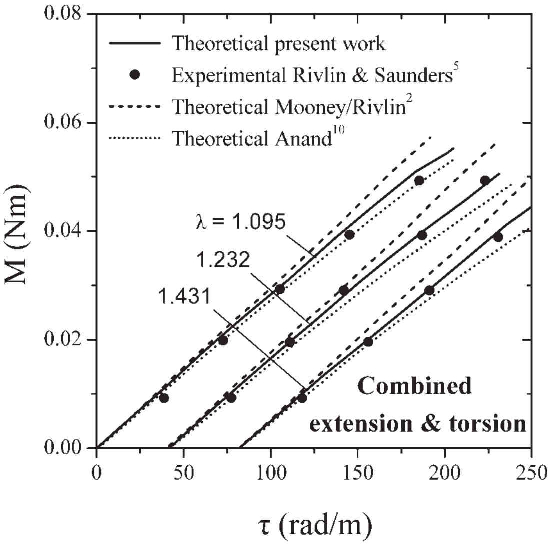

Figure 5 Twisting moment versus twisting angle per unit length for three different values of extension.

comparisons with the experimental data.[5] The material used is the same with the one used for the estimation of the material parameters in the previous section. Figure 3 presents, the variation of the twisting moment with the twisting angle per unit length for simple torsion ($\lambda = 1$). Figure 4 illustrates, the behavior of the axial force with respect to the square of twisting angle per unit length for simple torsion too. Figure 5 depicts the torque variation, as a function of the twisting angle per unit length for various values of stretch ($\lambda > 1$, combined extension and torsion). In all the above figures, the theoretical Mooney/Rivlin's[2] and Anand's[10,11] solutions are also included for comparative reasons. Because the torsional couple, M, for simple torsion is almost linear in the range of the applied angle of twist, t, all models follow the experimental data sufficiently. However, in the case where the cylinder is also under extension the proposed model is superior that of the others. In this case, the experimental points do not follow a linear dependence, and therefore the Mooney/Rivlin's as well as the Anand's solution fail to safely predict the experimental evidence. The proposed theory approaches the experimental points better than the other solutions published in the past and seems to permit the safe prediction of hyperelastic behavior of rubber for complicated twisting problems.

CONCLUSIONS

The logarithmic strain energy approach proposed in a previous work[7] has been extended to deal with the problems of rubber like materials under combined extension and torsion and therefore to prove its good performance for more complicated states of deformation. One more parameter has been used in this work to offer enhanced predictions for larger values of strain. Three of the material parameters k, h, and α have been determined by fitting corresponding simple tension and compression experimental data whereas the parameter g has been taken equal to $E/9$. Using the parameters arisen from the fitting process the equations describing the twisting moment and axial force have been defined and afterwards evaluated analytically using a common math code. The results illustrated good agreement with other experimental and theoretical solutions available in the literature. The computation of stresses inside the rubber material under combined torsion and extension using the proposed strain energy function in conjunction with the finite element method could be implemented in a future work.

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